Atiyah–Singer Index Theorem

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Abstract

The Atiyah-Singer index theorem is a seminal result in geometry and topology with many applications. This work presents an exposition on the Atiyah–Singer index theorem and related topics. In the first section, we survey the precursors of index theorems across mathematics. In the second section, we develop the requisite elements, including K-theory and Dirac operators, to state the index theorems. In the third section, we outline proofs of the theorems.

1 Motivations and Examples

There are motivations of the index theorem from various fields across mathematics, including algebraic topology, algebraic geometry and analysis. We discuss these following Freed [6].

1.1 Chern–Weil Theory and Chern–Gauss–Bonnet Theorem

1.1.1 Connection and Curvature

Let $E \to M$ be a (real or complex) vector bundle. A *connection* $\nabla = \nabla^E$ on E is a linear map

$$\nabla \colon \Gamma(M, E) \to \Omega^1(M, E) = \Gamma(M, T^*M \otimes E),$$

satisfying the Leibniz rule

$$\nabla(fs) = f\nabla s + s \otimes df.$$

A connection on E naturally induces connections on its dual E^*. Connections on two bundles induce a connection on their tensor product.

For a Riemannian manifold, there is a special connection called the Levi-Civita connection on TM.

The *curvature form* of the connection ∇ on E is the $\mathrm{End}(E)$-valued 2-form $R \in \Omega^2(M, \mathrm{End}(E))$ such that

$$R(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s, \quad X, Y \in \Gamma(TM), s \in \Gamma(E).$$

1.1.2 Chern–Weil Theory

For any complex vector bundle $E \to M$ with a connection ∇, Chern–Weil theory gives differential form representatives of characteristic classes from the curvature tensor $R \in \Omega^2(M, \mathrm{End}(E))$.

The earliest constructions are the Chern classes

$$c(\nabla) = \sum_k c_k(\nabla) := \left[\det\left(1 + \frac{iR}{2\pi}\right)\right] = \left[\exp \operatorname{tr} \ln\left(1 + \frac{iR}{2\pi}\right)\right]$$

and the Chern character

$$\mathrm{ch}(\nabla) = \sum_k \mathrm{ch}_k(\nabla) := \left[\operatorname{tr} \exp\left(\frac{iR}{2\pi}\right)\right].$$

Further examples that we will consider include the $\widehat{A}$-genus[1]

$$\widehat{A}(\nabla) = \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)$$

[1] This formula of the $\widehat{A}$-genus is for geometers; for topologists, R should be replaced with $R/(2\pi i)$.

and the Todd class

$$\mathrm{Todd}(\nabla) = \det\left(\frac{R}{\exp(R) - 1}\right),$$

which will play important roles in the Atiyah–Singer index theorem.

These classes, whose construction involves the curvature tensor, are actually independent of the choice of connection.

1.1.3 Chern–Gauss–Bonnet Theorem

The Chern–Gauss–Bonnet Theorem is an example along the path of Chern–Weil theory showing that topological invariants can be extracted from geometrical data. It states that, for M a compact orientable $2n$-dimensional Riemannian manifold,

$$\chi(M) = \int_M \frac{1}{(2\pi)^n}\,\mathrm{Pf}(R),$$

where Pf is the Pfaffian, and the class $[\mathrm{Pf}(R)]$ is the *Euler class* of TM. (If we regard R as a matrix with coefficients in evenly graded differential forms, then $\mathrm{Pf}(R)$ is well-defined since the Pfaffian is invariant under conjugation by matrices in $SO(n)$. It should be noted however that Pf is not an invariant polynomial.)

The Euler characteristic $\chi(M)$ is topological, and does not depend on the Riemannian metric.

1.2 Riemann–Roch Theorems

Enumerative problems in algebraic geometry often lead to topological invariants; the Riemann–Roch theorem is one example. On a smooth projective curve $X/\mathbb{C}$ with a divisor D,

$$\underbrace{\dim \mathcal{L}(D) - \dim \mathcal{L}(K - D)}_{\text{“analytic”}} = \underbrace{\deg(D) - g + 1}_{\text{“topological”}},$$

where $\mathcal{L}(D)$ is the space meromorphic functions having pole of order $\leq \mathrm{ord}_x D$ at each point $x \in X$.

Meromorphic functions are solutions to an *elliptic* PDE, i.e. we have $\mathcal{L}(D) = \ker \bar{\partial}$.

An alternative formulation of the Riemann–Roch Theorem is as follows. The *Euler characteristic* $\chi(X, \mathcal{O}(D))$ equals the dimension of $H^0(X, \mathcal{O}(D))$ minus the dimension of $H^1(X, \mathcal{O}(D))$. This can be computed as

$$\chi(X, \mathcal{O}(D)) = \deg(D) - g + 1,$$

indicating potential generalization to varieties and vector bundles of arbitrary dimension.

1.2.1 Hirzebruch–Riemann–Roch Theorem

Let X be a nonsingular projective variety of complex dimension n and $V \to X$ a holomorphic vector bundle.

The *Euler characteristic* of V is the alternating sum

$$\chi(X, V) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, V).$$

The Hirzebruch–Riemann–Roch (HRR) theorem states that

$$\underbrace{\chi(X, V)}_{\text{“analytic”}} = \underbrace{\mathrm{Todd}(X)\,\mathrm{ch}(V)[X]}_{\text{“topological”}},$$

where $\mathrm{Todd}(X)$ is the *Todd class* of X, and $\mathrm{ch}(V)$ is the *Chern character* of V.

The Euler characteristic can be thought of as a Chern character, namely $\mathrm{ch}(\sum(-1)^q H^q(X, F)) \in K(\mathrm{pt}) \simeq \mathbb{Z}$.

1.2.2 Grothendieck–Riemann–Roch Theorem

Grothendieck's dictum in developing algebraic geometry is to do geometry over a base, not just over a point. One result following this principle is the Grothendieck–Riemann–Roch (GRR) theorem. This theorem also marks Grothendieck's first introduction of *K-theory* in history. Its name derives from the German word *Klasse*, meaning “class”.

Let X be a smooth algebraic variety. Define $K(X)$ to be the free abelian group generated by coherent algebraic sheaves on X, modulo the equivalence $F \sim F' + F''$ for every short exact sequence $0 \to F' \to F \to F'' \to 0$. We will introduce K-theory in more detail in the next section. For now, let us emphasize one important fact: a coherent sheaf F on a smooth variety admits a *finite resolution by locally free sheaves* (i.e. *vector bundles*)

$$0 \to E_n \to \cdots \to E_1 \to E_0 \to F \to 0.$$

One can replace “coherent algebraic sheaves” by “holomorphic vector bundles”.

Let $f\colon X \to Y$ be a morphism of *smooth* varieties. For a coherent sheaf F on X, we define the *K-theoretic pushforward* $f_k\colon K(X) \to K(Y)$ by

$$f_k(F) = \sum_q (-1)^q R^q f_*(F).$$

When $Y = \mathrm{pt}$ the pushforwark f_k reduces to the *Euler characteristic*.

Now assume $f\colon X \to Y$ is *proper*. The GRR theorem states that for $\alpha \in K(X)$,

$$\mathrm{ch}(f_k\alpha)\,\mathrm{Todd}(Y) = f_*\big(\mathrm{ch}(\alpha)\,\mathrm{Todd}(X)\big).$$

One perspective is that the Todd class essentially "measures the lack of commutativity of push-forwards and the Chern character"; the required "correction factors" depend only on X and Y. If we take $Y = \mathrm{pt}$, the left-hand side becomes $\chi(X, \alpha)$, and the GRR theorem reduces to the HRR theorem.

1.2.3 Hirzebruch Signature Theorem

Let X be a real $4k$-dimensional closed oriented manifold. Its *signature* $\mathrm{Sing}(X)$ is the signature of the cup pairing on $H^{2k}(X; \mathbb{R})$.

We define the L-class in terms of the Chern roots y_i,

$$L(X) = \prod_{i=1}^{2k} \frac{y_i}{\tanh y_i}.$$

The Hirzebruch signature theorem states that

$$\underbrace{\mathrm{Sign}(X)}_{\text{"analytic"}} = \underbrace{L(X)[X]}_{\text{"topological"}}.$$

Hirzebruch showed that both sides are invariant under oriented bordism, and therefore it suffices to verify the formula on $\mathbb{C}P^{2n}$.

1.3 Analysis

We discuss some motivations for the index theorems in analysis. The index of an elliptic operator is constant as the operator is varied smoothly. It relates solvability (of PDEs) to topology. Detailed definitions are given in the next section.

1.3.1 Elliptic Operators and the Fredholm Index

The *Fredholm index* of an elliptic operator D is

$$\mathrm{ind}\, D = \dim\ker - \dim\mathrm{coker}\, D.$$

An elliptic operator is Fredholm, i.e. has finite-dimensional kernel and cokernel.

The Fredholm index of an elliptic operator is a deformation invariant, and depends only on the *homotopy class* of the symbol.

1.3.2 De Rham

On a closed Riemannian manifold X we have an elliptic operator

$$d + d^* \colon \Omega^{\text{even}}(X) \to \Omega^{\text{odd}}(X)$$

whose analytic index is $\operatorname{ind}(d + d^*) = \chi(X)$.

The kernel of $d + d^*$ consists of *harmonic forms*, and the above formula follows from Hodge theory. Note that $(d + d^*)^2 = dd^* + d^*d$ is the Laplacian on differential forms.

If X is $4k$-dimensional, we have the *signature operator*

$$d + d^* \colon \Omega^+(X, \mathbb{C}) \to \Omega^-(X, \mathbb{C})$$

whose index is $\operatorname{ind}(d + d^*) = \operatorname{Sign}(X)$, the *signature* of X.

This operator comes from a different $\mathbb{Z}/2\mathbb{Z}$-grading than the previous one.

Again the above formula follows from Hodge theory.

1.3.3 Dolbeault

On a Kähler manifold X with a holomorphic vector bundle $V \to X$, we have an elliptic operator

$$\bar\partial + \bar\partial^* \colon \Omega^{0,\text{even}}(X, V) \to \Omega^{0,\text{odd}}(X, V)$$

whose analytic index is

$$\operatorname{ind}(\bar\partial + \bar\partial^*) = \chi(X, V).$$

2 Statements

2.1 First Statement

In the first statement we define the topological index of an elliptic operator through the following steps.

$$\begin{array}{ccc} \text{Elliptic operators} & & \\ \sigma\downarrow & \searrow & \\ K(TX) & \xrightarrow[\text{t-ind}]{} & \mathbb{Z} \end{array}$$

We will explain the symbol σ, and then introduce K-theory to define the map t-ind.

2.1.1 Differential Operators and the Symbol

Definition 1 (differential operators)**.** Let E, F be vector bundles over the same manifold X. A *differential operator* from (the space of local sections of) E to F is a linear operator $D\colon \Gamma(E) \to \Gamma(F)$ which can be locally expressed as

$$D = \sum_{|\alpha|\leq m} A_\alpha(x)\partial^\alpha,$$

where α is a multiple index and $A_\alpha(x)$ is a linear map $E_x \to F_x$. The integer m is called the *order* of D (assuming $A_\alpha(x) \neq 0$ for some α with $|\alpha| = m$). The *symbol* of D is a bundle map $\sigma(D)\colon \operatorname{Sym}^m T^*X \otimes E \to F$ given by

$$\sigma(D)(\xi^\alpha) = \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha.$$

If we denote by $\pi^*\colon T^*X \to X$ the cotangent bundle, then $\sigma(D)$ can also be regarded as a bundle map $\pi^*E \to \pi^*F$ over T^*X, which restricts to $A_\alpha(x)\xi^\alpha$ on the point $(x, \xi) \in T^*X$.

2.1.2 K-theory

After Grothendieck introduced K-theory for projective algebraic varieties, Atiyah and Hirzebruch constructed a topological analog, $K(X)$, defined for *compact* (or generally, locally compact) spaces X.

Atiyah's philosophy is that K-theory is more elementary than cohomology. K-theory, based on linear algebra, is most compatible with linear differential operators.

We first review the basics of K-theory as presented by Atiyah [2].

Definition 2 (K-theory)**.** Let X be a compact space. Denote by $\mathrm{Vect}(X)$ the set of isomorphism classes of (complex) vector bundles over X, which is a monoid under direct sum. Define $K(X)$ to be the universal abelian group associated to $\mathrm{Vect}(X)$.

Every element of $K(X)$ is of the form $[E] - [F]$, where $[E]$ and $[F]$ are isomorphism classes of vector bundles over X.

The K-group is contravariant. For a compact space X with a distinguished basepoint $i\colon x_0 \to X$, define the reduced K-group $\widetilde{K}(X)$ to be the kernel of the map

$$i^*\colon K(X) \to K(x_0).$$

At times we need to consider vector bundles over the total space of other vector bundles, which require a generalization of the definition to locally compact spaces.

Definition 3 (K-theory with compact support)**.** Let X be a locally compact space. Consider the one-point compactification $\widetilde{X}$ and define the *K-theory with compact support* to be

$$K(X) := \widetilde{K}(\widetilde{X}).$$

Definition 4 (complex and support)**.** A *complex* of vector spaces over X is a sequence of bundle morphisms

$$0 \longrightarrow E_0 \xrightarrow{\alpha_1} E_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} E_k \longrightarrow 0$$

with $\alpha_{i+1} \circ \alpha_i = 0$ for $1 \le i < k$. At any point $x \in X$, we thus get a complex of vector spaces

$$0 \longrightarrow (E_0)_x \xrightarrow{\alpha_1} (E_1)_x \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} (E_k)_x \longrightarrow 0.$$

The *support* of a complex of vector spaces is the set of points where it fails to be exact.

Proposition 1. Let X be a locally compact space. Let $C(X)$ denote the set of homotopy classes of complexes over X with compact support, and let $C_\varnothing(X)$ be the subset consisting of exact complexes, i.e. complexes with empty support. Then

$$K(X) \simeq C(X)/C_\varnothing(X).$$

Proposition 2. If X is compact, then a complex $E_\bullet$ corresponds to the element

$$\chi(E_\bullet) = \sum_i (-1)^i [E_i] \in K(X),$$

where the map $\chi\colon C(X) \to K(X)$ is called the *Euler characteristic*.

Now we briefly discuss Grothendieck's work, described in [3], which predates that of Atiyah–Hirzebruch. Grothendieck defined two different K-groups, one using vector bundles (denoted by $K^0(X)$) and the other using coherent sheaves (denoted by $K_0(X)$). The group $K^0(X)$ forms a ring under tensor product, and $K_0(X)$ becomes a module over $K^0(X)$. However the two K-groups are isomorphic when X is non-singular.

2.1.3 Elliptic Operators and the Symbol Class

Definition 5. An *elliptic operator* from E to F is a differential operator D whose symbol $\sigma(D)$ gives an *isomorphism* $\pi^*E \to \pi^*F$ outside the zero section of T^*X; i.e. the map $\sigma(D)(\xi^\alpha)$ is a linear isomorphism $E_x \to F_x$ whenever $\xi \neq 0$.

Let D be an *elliptic operator* from E to F. Consider the cotangent bundle $\pi\colon T^*X \to X$; denote by π^*E the pullback of E to T^*X. The *symbol* of D is a map

$$\sigma(D)\colon \pi^*E \to \pi^*F,$$

which is an isomorphism on $T^*X \setminus 0$, and thus (by proposition 1) defines a class

$$[\sigma(D)] \in K(T^*X),$$

called the *symbol class* of D.

There is a general notion of an *elliptic complex*, which is a complex $(E_\bullet, D_\bullet)$ of differential operators

$$\Gamma(E_0) \xrightarrow{D_1} \Gamma(E_1) \xrightarrow{D_2} \Gamma(E_2) \longrightarrow \cdots \xrightarrow{D_k} \Gamma(E_k)$$

such that the sequence of symbols

$$0 \longrightarrow \pi^*E_0 \xrightarrow{\sigma(D_1)} \pi^*E_1 \xrightarrow{\sigma(D_2)} \pi^*E_2 \longrightarrow \cdots \xrightarrow{\sigma(D_k)} \pi^*E_k \longrightarrow 0$$

is *exact* outside the zero section of T^*X.

2.1.4 Analytic Index

An important result which allows the definition of the analytic index of an elliptic complex is the following.

Proposition 3. The homology groups of any elliptic complex are finite-dimensional.

Definition 6 (analytic index)**.** The *analytic* index of an elliptic complex $(E_\bullet, D_\bullet)$ is its Euler characteristic,

$$\text{a-ind}(E_\bullet, D_\bullet) = \chi(E) = \sum (-1)^i \dim H^i(E).$$

2.1.5 K-theoretic Thom Isomorphism

For a vector bundle $\pi\colon E \to X$ over a compact space X, consider the map $\alpha\colon \pi^* \wedge^\bullet E \to \pi^* \wedge^{\bullet+1} E$ sending a pair (v, w) $(v \in E, w \in \wedge^k E_{\pi(v)})$ to $(v, v \wedge w)$, giving the *exterior complex*

$$0 \longrightarrow \pi^* \wedge^0 E \xrightarrow{\alpha} \pi^* \wedge^1 E \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \pi^* \wedge^n E \longrightarrow 0$$

which defines an element $\lambda_E \in K(E)$.

Theorem 4 (Thom isomorphism)**.** If V is a complex vector bundle over a compact space X, then the *Thom map*

$$\phi\colon K(X) \to K(E), \quad u \mapsto \lambda_E \pi^* u$$

is an isomorphism.

If we denote by $i\colon X \to E$ the zero section, then by proposition 2 we have

$$i^*\phi(u) = \Big(\sum_q (-1)^q \wedge^q E \Big) \cdot u. \tag{1}$$

2.1.6 Topological Index

Following Lawson and Michelsohn [8], we define the topological index using K-theory.

First we introduce a K-theoretic pushforward map.

[2]

Definition 7. For an embedding $X \hookrightarrow Y$ of real manifolds, suppose that the normal bundle N to X is equipped with a complex structure. Then we can define a natural mapping

$$f_!\colon K(X) \to K(Y)$$

by taking the Thom isomorphism $K(X) \to K(N)$ followed by the map $K(N) \to K(Y)$, obtained by identifying N with a regular neighborhood of X in Y.

[2]Thie map resembles the map $f_k = \sum_q (-1)^q R^q f_*$ discussed in section 1.2.2, but the actual relationship remains unclear to the author.

Observe that for any proper embedding $f\colon X \to Y$, the normal bundle to the embedding $f_*\colon TX \to TY$, which equals the pullback of $N \oplus N \simeq N \otimes_{\mathbb{R}} \mathbb{C}$ to TX, has a canonical complex structure. Therefore we have a map

$$f_!\colon K(TX) \to K(TY).$$

Equation (1) implies

$$f^* f_!(u) = \Big(\sum (-1)^q \wedge^q (N \otimes_{\mathbb{R}} \mathbb{C})\Big) u.$$

Choose a smooth embedding $i\colon X \hookrightarrow \mathbb{R}^N$ into some Euclidean space, and consider the map

$$i_!\colon K(TX) \to K(T\mathbb{R}^N) \simeq K(\mathbb{C}^N).$$

Let $j\colon \mathrm{pt} \to \mathbb{C}^N$ be the inclusion of the origin. Note that by definition the map $j_!\colon K(\mathrm{pt}) \to K(\mathbb{C}^N)$ is equal to the Thom isomorphism ϕ (for the $\mathbb{C}^N$-bundle over a point).[3]

Definition 8. The *topological index* is the composition[4]

$$K(TX) \xrightarrow{i_!} K(T\mathbb{R}^N) \xrightarrow{j_!^{-1}} K(\mathrm{pt}) \simeq \mathbb{Z}.$$

The topological index can also be defined in terms of characteristic classes and cohomology,

$$\begin{array}{ccccc} K(T^*X) & & & \overset{\text{t-ind}}{\searrow} & \\ \downarrow{\scriptstyle \mathrm{ch}} & & & & \\ H^*_{\mathrm{cpt}}(T^*X) & \xleftarrow[\phi]{} & H^*(X) & \xrightarrow[\mathrm{Todd}(X)]{} & H^*(X) \end{array}$$

where ϕ is the *Thom isomorphism*.

Recall that the analytic index of D depends only on the *homotopy class* of the symbol, that is, on the class $[\sigma(D)] \in K(T^*X)$. Therefore the analytic index is also a map $K(T^*X) \to \mathbb{Z}$.

Theorem 5 (Atiyah–Singer 1967)**.** The analytic index and the topological index

$$\text{a-ind}, \text{t-ind}\colon K(T^*X) \to \mathbb{Z}$$

are equivalent.

[3] This isomorphism is also related to Bott periodicity: it equals the map $K(T\mathbb{R}^N) \to K(S^{2N}) \simeq K(\mathrm{pt})$.

[4] We make the identification $T^*X \simeq TX$ using a Riemannian metric on X.

2.2 Second Statement

The second statement of the Atiyah–Singer index theorem gives a formula for the index of a special elliptic operator, called the *Dirac operator* D, in terms of the Chern character of the Clifford module and the $\widehat{A}$-genus of the manifold. For this we will need concepts from *supergeometry* and *spin geometry.* Supergeometry is roughly the geometry corresponding to $\mathbb{Z}/2\mathbb{Z}$-graded algebras, and spin geometry are used to construct Dirac operators.

2.2.1 Super Vector Spaces

We restrict our attention to vector spaces over $\mathbb{R}$ or $\mathbb{C}$. A *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space; that is, it decomposes into a direct sum of two vector spaces V_0 and V_1, called the *even* and *odd* parts respectively. Elements of V_i are called *pure* elements of degree i ($i \in \mathbb{Z}/2\mathbb{Z}$). The degree of a pure element x is denoted by $|x|$. We will also use the notation $V = V^+ \oplus V^-$ where V^+ is even and V^- is odd.

The category of super vector spaces can be made into a symmetric monoidal category with the tensor product

$$(X \otimes Y)_i = \bigoplus_{j+k=i} X_j \otimes Y_k \quad (i, j, k \in \mathbb{Z}/2\mathbb{Z})$$

and the braiding $B_{X,Y} \colon X \otimes Y \to Y \otimes X$ given by

$$B_{X,Y}(x \otimes y) = (-1)^{|x|\cdot|y|} y \otimes x,$$

where $x \in X$ and $y \in Y$ are pure elements.

The inner Hom-object of this category is given by

$$\big(\underline{\mathrm{Hom}}(X, Y)\big)_i = \bigoplus_{j+i=k} \mathrm{Hom}(X_j, Y_k) \quad (i, j, k \in \mathbb{Z}/2\mathbb{Z}).$$

In other words, the even part of $\underline{\mathrm{Hom}}(X, Y)$ consists of linear maps $X \to Y$ that preserve the grading, while the odd part contains those that reverse the grading.

A *super algebra* is a super vector space A with a morphism $A \otimes A \to A$ satisfying certain axioms (associativity, unit, etc.); that is, an algebra in the category of super vector spaces. An *action* of a super algebra A on a super vector space is a map $A \otimes E \to E$, or equivalently a super algebra homomorphism $A \to \underline{\mathrm{End}}(E)$, according to a property of the inner Hom-object.

When working with superalgebras, one must keep in mind that whenever two elements exchange positions, a sign may appear depending on their degrees[5]. An example is the supercommutator, defined by

$$[x, y] = xy - (-1)^{|x|\cdot|y|}yx.$$

2.2.2 Clifford Algebras and Clifford Modules

Definition 9 (Clifford algebra)**.** Let V be a vector space with a quadratic form Q. The *Clifford algebra* $\mathrm{Cl}(V) = \mathrm{Cl}(V, Q)$ is the algebra generated by V with the relations

$$x^2 = -Q(x)\,(x \in V).$$

The Clifford algebra $\mathrm{Cl}(V)$ can be constructed as the quotient of the tensor algebra $\mathscr{T}V$ by an evenly graded ideal,

$$\mathrm{Cl}(V) = \mathscr{T}V\big/\big(\{x \otimes x + Q(x)\colon x \in V\}\big),$$

and thus $\mathrm{Cl}(V)$ has a $\mathbb{Z}/2\mathbb{Z}$ grading inherited from $\mathscr{T}V$, making it a super algebra.

Note that for two elements $x, y \in V$ satisfying the relation $Q(x, y) = 0$, denoted $x \perp y$, we have $xy = -yx$; that is, x and y anticommute. Therefore the space $\mathrm{Cl}(V)$ is isomorphic to the *exterior algebra* $\wedge V$ as a super vector space (but clearly not as a super algebra) through the map $\mathbf{c}\colon \wedge V \to \mathrm{Cl}(V)$, given by

$$\mathbf{c}\colon e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r} \mapsto e_{i_1} e_{i_2} \cdots e_{i_r}$$

for any orthogonal basis $\{e_i\}$ of V. In fact, this isomorphism is natural, and in particular it is independent of the orthogonal basis chosen. The inverse of $\mathbf{c}$ is called the *symbol map*

$$\sigma\colon \mathrm{Cl}(V) \to \wedge V.$$

Definition 10 (Clifford module)**.** A *Clifford module* is a super vector space $E = E^+ \oplus E^-$ equipped with an action of $\mathrm{Cl}(V)$, i.e. a super algebra homomorphism $\mathrm{Cl}(V) \to \underline{\mathrm{End}}(E)$.

Observe that the action of $\mathrm{Cl}(V)$ on a Clifford module E can be specified by a map $c\colon V \to \underline{\mathrm{End}}(E)^-$ satisfying

$$[c(x), c(y)] = c(x)c(y) + c(y)c(x) = -2Q(x, y).$$

[5]This point is demonstrated most clearly through the categorical approach, as described in [5]. The book [5] also contains a lecture by E. Witten describing a path integral proof of the Atiyah–Singer index theorem.

Example 1 (exterior algebra as Clifford module)**.** The *exterior algebra* $\wedge V$ is a Clifford module with the action as follows. Denote by $v^\flat = Q(v, -)$ the dual of a vector $v \in V$, and denote by $\iota_{v^\flat} \colon \wedge^\bullet V \to \wedge^{\bullet-1} V$ the contraction. Consider the map $c \colon V \to \underline{\mathrm{End}}(E)^-$ given by

$$c(v)\alpha = v \wedge \alpha - \iota_{v^\flat}\alpha.$$

Then it can be checked that $c(v)c(w) + c(w)c(v) = -2Q(v, w)$.

For a Riemannian manifold M there is a quadratic form on TM, so that the construction of the Clifford algebra yields a *bundle of Clifford algebras* $\mathrm{Cl}(TM)$.

The symbol map from Clifford algebras to exterior algebras gives an isomorphism

$$\sigma^{-1} \colon \Omega^\bullet(E) \simeq \Gamma(\mathrm{Cl}(TM) \otimes E).$$

A Bundle E of Clifford modules is a Hermitian super vector bundle with an action

$$\mathrm{cl} \colon \mathrm{Cl}(TM) \otimes E \to E.$$

They are needed to introduce the *Dirac operators*, whose squares are equal to Laplacians.

2.2.3 Spin Manifolds and Spinor bundles

The Lie group $\mathrm{Spin}(n)$ is a double cover of $SO(n)$, which can be defined using a Clifford algebra. The following definition of $\mathrm{Spin}(n)$, taken from [4], is less commonly seen but more concise. We start with the Lie algebra.

Proposition 6. The space $C^2(V) := \mathbf{c}(\wedge^2 V)$, i.e. the vector space spanned by the 2-vectors $xy\,(x, y \in V, x \perp y)$, is a Lie algebra under the commutator $[-,-]$ in $\mathrm{Cl}(V)$. Moreover it is isomorphic to $\mathfrak{so}(V)$ through the map $\tau \colon C^2(V) \to \mathfrak{so}(V)$,

$$\tau(a) \cdot v = [a, v].$$

Definition 11 (spin group)**.** The group $\mathrm{Spin}(V)$ is the group obtained by exponentiating the Lie algebra $C^2(V)$ inside the Clifford algebra $\mathrm{Cl}(V)$.

Definition 12 (spin structure)**.** A *spin structure* on a manifold M is a reduction of the structure group of its principal $SO(n)$-bundle $P_{\mathrm{SO}}(M)$ through $\mathrm{Spin}(n) \to SO(n)$. In other words, a spin structure is a double cover $P_{\mathrm{Spin}}(M) \to P_{\mathrm{SO}}(M)$, where $P_{\mathrm{Spin}}(M)$ is a principal $\mathrm{Spin}(n)$-bundle, and the double cover is compatible with the actions of $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$. A manifold with a spin structure is called a *spin manifold*.

Equivalently a spin structure is a $\mathbb{Z}/2$-graded irreducible $\mathrm{Cl}(TM)$-$\mathrm{Cl}(\mathbb{R}^n)$ bimodule.

If V is a $2k$-dimensional quadratic vector space, then there is a unique $\mathrm{Cl}(V)$-module S such that

$$\mathrm{End}(S) \simeq \mathrm{Cl}(V) \otimes \mathbb{C}.$$

Using this on a spin manifold we define a special Clifford module called the *spinor bundle*

$$\not{S} := \mathrm{Spin}(M) \otimes_{\mathrm{Spin(n)}} S.$$

2.2.4 Clifford Superconnections and Curvature

A *superconnection* on a super vector bundle is an odd map

$$\mathbb{A} \colon \Omega^*(M, E) \to \Omega^*(M, E)$$

satisfying the Leibniz rule

$$\mathbb{A}(\alpha \wedge s) = d\alpha \wedge s + (-1)^{|\alpha|} \alpha \wedge \mathbb{A}(s) \, (\alpha \in \Omega^*(M), s \in \Omega^*(M, E)).$$

A *Clifford superconnection* is one compatible with the Levi-Civita connection ∇ on TM:

$$[\mathbb{A}, \mathrm{cl}(v)] = \mathrm{cl}(\nabla v) \, (v \in \Gamma(M, TM)).$$

Similar to the classical case, the curvature is related to the square of connection. Computation shows

$$[\mathbb{A}^2, \mathrm{cl}(v)] = \mathrm{cl}(\nabla^2 v) = \mathrm{cl}(Rv) = [R^{\mathcal{E}}, \mathrm{cl}(v)],$$

which justifies the following result.

Proposition 7. For $\mathbb{A}$ a Clifford superconnection on E, we have

$$\mathbb{A}^2 = R^{\mathcal{E}} + F^{\mathcal{E}/S},$$

where $R^{\mathcal{E}} \in \Omega^2(M, \mathrm{End}\,\mathcal{E})$ is (a Clifford version of) the curvature, and $F^{\mathcal{E}/S} \in \Omega^2(M, \mathrm{End}_{\mathrm{Cl}(TM)}(\mathcal{E})$ is called the *twisting curvature*.

On a *spin manifold*, all Clifford bundles $\mathcal{E}$ are twists of the *spinor bundle* $\not{S}$, i.e. $\mathcal{E} = \not{S} \otimes W$. The twisting curvature is then just the curvature of W.

2.2.5 Dirac Operators

Dirac operators play an important role in the index theorem. In analogy to the classical Dirac operator

$$D\colon \Gamma(M,E) \xrightarrow{\nabla} \Omega^1(M,E) \xrightarrow{\text{metric}} \Gamma(M, TM\otimes E) \xrightarrow{\text{cl}} \Gamma(M,E)$$

we define the Dirac operator $D_{\mathbb{A}}$ associated to a Clifford superconnection $\mathbb{A}$ by

$$D_{\mathbb{A}}\colon \Gamma(M,E) \xrightarrow{\mathbb{A}} \Omega^*(M,E) \xrightarrow{\sigma^{-1}} \Gamma(M, \mathrm{Cl}(TM)\otimes E) \xrightarrow{\text{cl}} \Gamma(M,E).$$

Dirac operators were originally motivated by the search for a first-order differential operator whose square equals the Laplacian.

Proposition 8 (Lichnerowicz)**.** The square of a Dirac operator D satisfies

$$D_{\mathbb{A}}^2 = \triangle^{\mathbb{A}} + \mathrm{cl}(F^{\mathcal{E}/S}) + \frac{r_M}{4},$$

where r_M is the scalar curvature of M.

For the spinor bundle $\mathcal{E} = \$$, the twisting $F^{\mathcal{E}/S}$ vanishes so that $\not{D}^2 = \triangle^{\$} + r_M/4$.

2.2.6 $\widehat{A}$-genus and Chern Character

For a Riemannian manifold M with Ricci curvature R we define the $\widehat{A}$-genus to be

$$\widehat{A}(M) = \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right).$$

Definition 13 (Relative Chern character)**.** We define the *relative Chern character* of a Clifford bundle $\mathcal{E}$ as

$$\mathrm{ch}(\mathcal{E}/S) = \mathrm{str}_{\mathcal{E}/S}\exp(-F^{\mathcal{E}/S}).$$

$\mathrm{ch}(\mathcal{E}/S)$ is a closed form; its class is independent of the choise of the Clifford superconnection.

Theorem 9 (Atiyah–Singer)**.** The index of a Dirac operator D of a Clifford module $\mathcal{E}$ is

$$\mathrm{ind}(D) = \int_M \widehat{A}(M)\,\mathrm{ch}(\mathcal{E}/S),$$

For the Clifford module $\mathcal{E} = \wedge^* T^*M$ with the Dirac operator $D = d + d^*$ this reduces to the Chern–Gauss–Bonnet theorem.

For the Clifford module $\mathcal{E} = \wedge^{0,*}T^*M \otimes V$ with the Dirac operator $D = \bar\partial + \bar\partial^*$ this reduces to the Hirzebruch–Riemann–Roch theorem.

2.3 Familiy Index Theorem

In the same sense that the Grothendieck–Riemann–Roch theorem generalizes the Riemann–Roch theorem, there is a generalization of the Atiyah–Singer index theorem called the family index theorem.

Denote by $\mathcal{F}$ the space of Fredholm operators on a Hilbert space H. To a continuous family $X \to \mathcal{F}$ of Fredholm operators, we can assign an analytic index in $K(X)$, which when $X = \mathrm{pt}$ (i.e. for a single operator) reduces to an integer.

2.4 Equivariant Index Theorem

Another important generalization of the index theorem considers the equivariant case, where there exists an action of a compact Lie group.

2.4.1 Equivariant K-theory

Equivariant K-theory, an equivariant analog of K-theory, plays an essential role in formulating the equivariant index theorem. Equivariant K-theory also appears in the axioms for the index proposed by Atiyah and Singer [1].

Definition 14 (G-vector bundles)**.** For a Lie group G and a topological space X, a *G-vector bundle* on X is a vector bundle $\pi : E \to X$ equipped with an action of G on both E and X satisfying the following conditions:

- The G-action commutes with the projection π.
- For each $g \in G$ and $x \in X$, the map $E_x \to E_{gx}$ induced by g is a linear transformation.

Let G be a compact Lie group. By considering G-vector bundles on X, we can define a ring $K_G(X)$. When G is the trivial group, $K_G(X)$ reduces to $K(X)$. If X is a point, a G-vector bundle on X is just a G-representation, and so $K_G(X) = K_G(\mathrm{pt})$ reduces to the representation ring $R(G)$ of G. For general spaces X, $K_G(X)$ is a module over $R(G)$.

2.4.2 G-Indices

As in the ordinary case, we can define the topological G-index by

$$\text{t-ind}_G \colon K_G(TX) \xrightarrow{i_!} K_G(TM) \xrightarrow{j_!^{-1}} K_G(\mathrm{pt}) \simeq R(G),$$

where $i\colon X \to M$ is an embedding of X into a finite-dimensional G-module M, and $j\colon \mathrm{pt} \to M$ is the inclusion of the origin.

An elliptic operator D from E to F is called *G-invariant* if it commutes with the G-actions on $\Gamma(E)$ and $\Gamma(F)$. The *analytic G-index* of D is defined by

$$[\ker D] - [\operatorname{coker} D] \in K_G(\mathrm{pt}) \simeq R(G).$$

This gives a map

$$\text{a-ind}_G\colon K_G(TX) \to R(G).$$

Theorem 10 (G-Index Theorem)**.** The topological and analytic G-indices

$$\text{t-ind}_G, \text{a-ind}_G\colon K_G(TX) \to K_G(\mathrm{pt}) \simeq R(G)$$

are equivalent.

3 Proofs

3.1 K-theory Method

In their joint paper [1], Atiyah and Singer present a proof of their index theorem in purely K-theoretic terms, while "no homology or cohomology is used".

Atiyah and Singer defines two "index homomorphisms"

$$\mathrm{ind}\colon K(T^*X) \to \mathbb{Z},$$

and uniquely characterize the index homomorphisms by axioms, thus showing that

$$\text{analytic index} = \text{topological index}.$$

3.2 Heat Kernel Method

The Heat flow provide intuition about the large and small time behavior of the heat operator $e^{-t\triangle}$. As $t \to \infty$, it converges to the projection onto $\ker \triangle$. As $t \to 0^+$ it converges to id.

We consider the operator e^{-tD^2} for D a Dirac operator. The key point is that the super trace $\mathrm{str}(e^{-tD^2})$ is independent of t. As $t \to 0$, we have an asymptotic expansion of $\mathrm{str}(e^{-tD^2})$ which gives

$$\mathrm{str}(e^{-tD^2}) = \mathrm{ind}(D).$$

3.2.1 Density Bundles

Definition 15. Let M be a manifold. The *s-density bundle* $|\Lambda|^s$ is the line bundle associated to TM via the representation $|\det|^{-1/2}\colon GL_n(\mathbb{R}) \to \mathbb{R}^\times$. In case $s = 1$ we call it the *density bundle* $|\Lambda|$.

We can integrate (compact-supported) sections of the density bundle $|\Lambda|$, even without orientation. Consequently, for a vector bundle E we have a pairing $\Gamma(E) \times \Gamma(E^* \otimes |\Lambda|) \to \mathbb{R}$; moreover if E is an Euclidean vector bundle, the there is a canonical inner product on $\Gamma(E \otimes |\Lambda|^{1/2})$.

Definition 16. Let E be a vector bundle on a Riemannian manifold M. A *generalized Laplacian* on E is a second-order differential operator H on E whose symbol is $\sigma(H)(x, \xi) = |\xi|^2$.

3.2.2 Heat Kernel

The heat kernel is the integral kernel of the heat semigroup, and also the fundamental solution to the heat equation.

On $\mathbb{R}^n$ with the Laplacian $\triangle = -\frac{\partial}{\partial x_1^2} - \cdots - \frac{\partial}{\partial x_n^2}$, the heat kernel is

$$K_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(x-y)^2}{4t}},$$

which satisfies (in the sense of distributions)

$$(\partial_t + \triangle_x) K_t(x,y) = \delta(x-y).$$

The heat kernel solves the heat equation in the sense that, for any compactly supported smooth function f on $\mathbb{R}^n$, the function

$$u_t(x) = \int_{\mathbb{R}^n} K_t(x,y) f(y)\, dy$$

satisfies $\lim_{t\to 0^+} u_t(x) = f(x)$, and $(\partial_t + \triangle) u_t(x) = 0$.

Analogously, we may define heat kernels on general manifolds.

Definition 17 (heat kernel)**.** Let E be a vector bundle over a Riemannian manifold M. Let H be a generalized Laplacian on $E \otimes |\Lambda|^{1/2}$. A *heat kernel* for H is a section $K_t(x,y)$ of $(E \otimes |\Lambda|^{1/2}) \boxtimes (E^* \otimes |\Lambda|^{1/2})$ over $\mathbb{R}_+ \times M \times M$, which is C^1 in $t \in \mathbb{R}_+$, C^2 in $x \in M$, and satisfies

$$(\partial_t + H_x) K_t(x,y) = \delta(x-y).$$

Theorem 11 (McKean–Singer)**.** For a Dirac operator D,

$$\operatorname{ind}(D) = \int_M \operatorname{str}(K_t(x,x))\, dx.$$

One consequence of this formula is that $\operatorname{ind}(D)$ is independent of the metric of M and the metric and connection of E, since the integral varies smoothly for a family, and the index has to be an integer.

3.2.3 Asymptotic Expansion

Getzler [7] provides a proof of the index theorem using the asymptotic expansion of the heat kernel at $t \to 0$.

Theorem 12 (Getzler)**.** We have the asymptotic expansion

$$K_t(x,x) \sim (4\pi t)^{n/2} \sum_{i=0}^{\infty} k_i(x) t^i$$

for $k_i \in \Gamma(\mathrm{Cl}_{2i}(TM) \otimes \mathrm{End}_{\mathrm{Cl}(TM)}(E))$, satisfying

$$\sigma(K) = \sum_{i=0}^{n/2} \sigma_{2i}(k_i) = \det{}^{1/2}\Big(\frac{R/2}{\sinh(R/2)}\Big) \exp(-F^{\mathcal{E}/S}).$$

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